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On the Initial Layer Solution of the Boltzmann Equation with Small Knudsen Number

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We extend our method of systematic removal of secular terms in a singular perturbation treatment of the Boltzmann equation with small Knudsen numbers to the initial layer. The requirement that the solution through the initial layer should connect smoothly to the normal solution removes an ambiguity noted in our previous paper. We show that removal of secular terms improves Grad's solution for the initial layer and reintroduces soundlike modes associated with higher moments, first found by Wang Chang and Uhlenbeck.

KEY WORDS: Boltzmann equation; "initial layer"; relaxation; singular perturbation.

1. INTRODUCTION

Our previous paper[1] was devoted to a discussion of the normal solution of the Boltzmann equation (B.E.) with small Knudsen number. That solution cannot be used to study the relaxation behavior in the initial layer, the boundary layer, and the shock wave layer. In the present paper, the initial layer solution will be discussed in detail. We again, for simplicity, consider only the case of planar geometry. The method can be generalized to more complicated geometries without essential difficulties.

In Ref. 1 we found that the Fourier transform in velocity space of the B.E. for the case of planar geometry can be written as

$$\frac{\partial\varphi}{\partial t}+i\mu\frac{\partial^2\varphi}{\partial k\,\partial\mu}+\frac{i(1-\mu^2)}{k}\frac{\partial^2\varphi}{\partial\mu\,\partial x}=\frac{1}{\varepsilon}J(\varphi,\varphi) \tag{1}$$

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where the generating function

$$\varphi = \varphi(x, k, \mu, t) \tag{2}$$

is the Fourier transform of the distribution function $f(x, v, v, t)$, and

$$J(\varphi, \varphi_1) = \int d\hat{u}' \, d\mathbf{k}' \, g(\hat{k} \cdot \hat{u}', k') \left\{ \varphi_1 \left[\frac{k}{2} (\hat{k} + \hat{u}') - \mathbf{k}' \right] \varphi \left[\frac{k}{2} (\hat{k} - \hat{u}') + \mathbf{k}' \right] - \varphi_1(\mathbf{k} - \mathbf{k}') \, \varphi(\mathbf{k}') \right\} \tag{3}$$

We use the same notation as that of Ref. 1 throughout.

We write the initial value for the generating function φ as

$$\varphi(t=0) = \rho_0 \exp[-\tfrac{1}{2}k^2\theta_0 - ik\mu c_0](1 + \psi_0) \tag{4}$$

where

$$\rho_0 = \rho(x, 0), \qquad c_0 = c(x, 0), \qquad \theta_0 = \theta(x, 0)$$

$$\psi_0 = \sum_{nl} b_{nl}^{(0)}(x, 0) \, e_{nl}, \qquad (n, l) = (2, 2), (3, 1),... \tag{5}$$

Here the quantities e_{nl} are defined as

$$e_{nl} = \frac{(-ik)^n}{n!} P_l(\mu), \qquad n = 0, 1,...; \quad l = n, n-2,..., 1, \text{ or } 0 \tag{6}$$

and $P_l(\mu)$ are Legendre polynomials. We stressed in Ref. 1 that the e_{nl} are eigenfunctions of the linearlized collision operator, and that the normal solution

$$\varphi_n = \varphi_0[1 + \xi(x, k, \mu, t)] \tag{7}$$

satisfies the B.E. (1)

$$\frac{\partial \varphi_n}{\partial t} + i\mu \frac{\partial^2 \varphi_n}{\partial k \, \partial x} + \frac{i(1-\mu^2)}{k} \frac{\partial^2 \varphi_n}{\partial \mu \, \partial x} = \frac{1}{\varepsilon} J(\varphi_n, \varphi_n) \tag{8}$$

but it does not satisfy the initial conduction (4) in general.

Assuming that the solution of Eq. (1) with the initial condition (4) consists of two parts

$$\varphi = \varphi_n + \varphi_i \tag{9}$$

where φ_n is the normal solution obtained in Ref. 1, we obtain from Eqs. (1), (8), and (9) that φ_i must satisfy the following equation:

$$\frac{\partial\varphi_i}{\partial t}+i\mu\frac{\partial^2\varphi_i}{\partial k\,\partial x}+\frac{i(1-\mu^2)}{k}\frac{\partial^2\varphi_i}{\partial\mu\,\partial x}$$
$$=\frac{1}{\varepsilon}\left[J(\varphi_i,\varphi_n)+J(\varphi_n,\varphi_i)+J(\varphi_i,\varphi_i)\right] \tag{10}$$

and the initial condition

$$\varphi_i(t=0)=\varphi(t=0)-\varphi_n(t=0) \tag{11}$$

In addition, φ_i should satisfy

$$\varphi_i(t\to\infty)=0 \tag{12}$$

which guarantees that φ becomes the normal solution φ_n outside the initial layer. We shall call φ_i the initial layer solution.

The initial layer solution and its connection to the normal solution was discussed by Grad,[2] using the moment method. Surprisingly, he did not find the soundlike modes pointed out already by Wang Chang and Uhlenbeck[4] to be associated with higer moments. These are sufficiently strongly damped that their physical significance is doubtful, but the conceptual discrepancy between Wang Chang and Uhlenbeck's work and that of Grad remains. In this paper we shall resolve this conflict: Grad's expansion will be shown to contain secular terms. When those terms are removed, the soundlike modes reappear.

Also, the ambiguity in the coefficients of the power expansions in Ref. 1 of the hydrodynamic variables is removed. We shall show that the initial layer solution can only connect smoothly to a normal solution in which the hydrodynamic variables are expanded in powers of ε, and with properly adjusted coefficients. If the hydrodynamic fields are expanded in powers of $\varepsilon^{N+1}=\varepsilon^2$, secular terms appear in the initial layer solution. This demonstrates the superiority of the expansion method of Ref. 1 over that proposed by Cercignani.[5]

Write the initial layer solution as

$$\varphi_i=\varphi_0\psi \tag{13}$$

where φ_0 is given in Ref. 1 as follows:

$$\varphi_0=\rho(x,t)\exp\left[-\tfrac{1}{2}k^2\theta(x,t)-ik\mu c(x,t)\right] \tag{14}$$

and

$$\psi = \psi(x, k, \mu, t)$$

From Eqs. (10) and (11) one finds that

$$\varepsilon \frac{\partial \psi}{\partial t} - \rho I(\psi) = \rho J'(\psi, \psi) + \rho \sum_{j=1}^{\infty} \varepsilon^j [J'(\psi, \xi^{(j)}) + J'(\xi^{(j)}, \psi)]$$
$$-\varepsilon [D_0 \psi + D_1(\psi) + D_2(\psi)] \tag{15}$$

$$\psi(t=0) = \sum_{nl} \left[b_{nl}^{(0)}(x, 0) - \sum_{j=1}^{\infty} \varepsilon^j a_{nl}^{(j)}(x, 0) \right] e_{nl} \tag{16}$$

where the symbols $I(\psi)$, $J'(\psi, \psi_1)$, $\xi^{(j)}$, D_0, $D_1(\psi)$, $D_2(\psi)$, and $a_{nl}^{(j)}$ were defined in Ref. 1.

In general, all the $b_{nl}^{(0)}(x, 0)$ do not vanish, so we have $\psi(t=0) \sim O(1)$. It is seen from Eq. (15) that

$$\partial \psi / \partial t \sim O(1/\varepsilon)$$

which shows the rapid evolution of ψ. In view of this, we introduce a new time scale

$$\tau = t/\varepsilon \tag{17}$$

and rewrite Eq. (15) as

$$\frac{\partial \psi}{\partial \tau} - \rho I(\psi) = \rho J'(\psi, \psi) + \rho \sum_{j=1}^{\infty} \varepsilon^j [J'(\psi, \xi^{(j)}) + J'(\xi^{(j)}, \psi)]$$
$$-\varepsilon [D_0 \psi + D_1(\psi) + D_2(\psi)] \tag{18}$$

Let

$$\psi = \sum_{j=0}^{\infty} \varepsilon^j \psi^{(j)} = \sum_{j=0}^{\infty} \varepsilon^j \sum_{nl} b_{nl}^{(j)}(x, \tau)\, e_{nl} \tag{19}$$

It is seen from Eqs. (5) and (19) that

$$\psi^{(0)}(\tau = 0) = \psi_0$$

Assume that

$$\frac{\partial \psi}{\partial \tau} = \sum_{j=0}^{\infty} \varepsilon^j \left(\frac{\partial \psi}{\partial \tau} \right)_j \tag{20}$$

and

$$\frac{\partial b_{nl}^{(j)}(x, \tau)}{\partial \tau} = \sum_{i=0}^{\infty} \varepsilon^i \beta_{nl}^{(j,i)} \tag{21}$$

where $\beta_{nl}^{(j,i)}$ depend on ρ, c, θ, and $b_{n'l'}^{(k)}$ (with $k < j$ and with $k = j$ in the case $n' < n$) and their space derivatives. Making use of Eqs. (19)–(21), we get

$$\left(\frac{\partial \psi}{\partial \tau}\right)_j = \sum_{nl} \left[\sum_{i=0}^{j} \beta_{nl}^{(i,j-i)}\right] e_{nl} \tag{22}$$

For Maxwell molecules, Eq. (18) reduces to

$$\begin{aligned}\frac{\partial \psi}{\partial \tau} - \rho I_{\mathrm{M}}(\psi) = \rho J_{\mathrm{M}}(\psi, \psi) + \rho \sum_{j=1}^{\infty} \varepsilon^j [J_{\mathrm{M}}(\psi, \xi^{(j)}) + J_{\mathrm{M}}(\xi^{(j)}, \psi)] \\ - \varepsilon[D_0 \psi + D_1(\psi) + D_2(\psi)]\end{aligned} \tag{23}$$

We shall consider mainly Maxwell molecules.

2. THE DOMINANT TERMS OF THE "INITIAL LAYER" SOLUTION

Substituting Eqs. (19) and (22) into Eq. (23), we find that the ε^0-order approximation to Eq. (23) is

$$\left(\frac{\partial \psi}{\partial t}\right)_0 - \rho I_{\mathrm{M}}(\psi^{(0)}) = \rho J_{\mathrm{M}}(\psi^{(0)}, \psi^{(0)}) \tag{24}$$

Equating the coefficients of e_{nl} on both sides of the above equation, we obtain

$$\beta_{nl}^{(0,0)} + \lambda_{nl} \rho b_{nl}^{(0)} = R_{nl}^{(0)} \tag{25}$$

where

$$R_{nl}^{(0)} = \rho \sum_{n'l'n''l''} b_{n'l'}^{(0)} b_{n''l''}^{(0)} h_{\mathrm{M}nl}^{n'l'n''l''} \tag{26}$$

and the coefficients $h_{\mathrm{M}nl}^{n'l'n''l''}$ have been defined in Ref. 1.

It should be noticed that $R_{nl}^{(0)}$ depends only on $b_{n'l'}^{(0)}$ for $n' < n$ because of the properties of $h_{\mathrm{M}nl}^{n'l'n''l''}$. Hence, we may determine all $b_{nl}^{(0)}$ from Eq. (25) one by one as long as $\beta_{nl}^{(0,0)}$ have been defined. In fact, the form of Eq. (25) is quite similar to that of the moment equations of the B.E. for Maxwell molecules in the spatially homogeneous case, where one can evaluate the time evolution of subsequent moments by solving ordinary differential equations.[6] If we retain only the first term on the right-hand side of Eq. (21), we find that Eq. (25) is exactly the spatially homogeneous B.E.

3. THE ε-ORDER APPROXIMATION

The ε-order approximation to Eq. (23) is

$$\left(\frac{\partial\psi}{\partial\tau}\right)_1 - \rho I_M(\psi^{(1)} = \rho[J_M(\psi^{(0)}, \psi^{(1)}) + J_M(\psi^{(1)}, \psi^{(0)})] + \rho[J_M(\psi^{(0)}, \xi^{(1)}) + J_M(\xi^{(1)}, \psi^{(0)})] - [D_{00}\psi^{(0)} + D_1(\psi^{(0)}) + D_2(\psi^{(0)})] \tag{27}$$

Equating the coefficients of e_{nl} on both sides of the above equation, we obtain

$$\beta_{nl}^{(1,0)} + \beta_{nl}^{(0,1)} + \lambda_{nl}\rho b_{nl}^{(1)} = -\frac{n(l+1)}{2l+3}\theta\frac{\partial b_{n-1,l+1}^{(0)}}{\partial x} - c\frac{\partial b_{nl}^{(0)}}{\partial x} - \frac{(n-l+2)\,l}{(n+1)(2l-1)}\frac{\partial b_{n+1,l-1}^{(0)}}{\partial x} + \tilde{\beta}_{nl}^{(1)} \tag{28}$$

where

$$\tilde{\beta}_{nl}^{(1)} = 2\rho\sum_{n'l'n''l''} b_{n'l'}^{(0)}(b_{n''l''}^{(1)} + a_{n''l''}^{(1)})\,h_{Mnl}^{n'l'n''l''} - \frac{1}{2}n(n-1)(n-2)\,\theta\frac{\partial\theta}{\partial x}\left(\frac{l}{2l-1}b_{n-3,l-1}^{(0)} + \frac{l+1}{2l+3}b_{n-3,l+1}^{(0)}\right) - n(n-1)\,\theta\frac{\partial c}{\partial x}\left[\frac{l(l-1)}{(2l-1)(2l-3)}b_{n-2,l-2}^{(0)} + \frac{2l(l+1)}{3(2l-1)(2l+3)}b_{n-2,l}^{(0)} + \frac{(l+1)(l+2)}{(2l+3)(2l+5)}b_{n-2,l+2}^{(0)}\right] - \frac{nl}{2l-1}\theta\frac{\partial b_{n-1,l-1}^{(0)}}{\partial x} - \frac{n}{2}\frac{\partial\theta}{\partial x}\left[\frac{l(n-l)}{2l-1}b_{n-1,l-1}^{(0)}\right.$$

$$+\frac{(l+1)(n+l+1)}{2l+3}b^{(0)}_{n-1,l+1}\Bigg]$$

$$-\frac{\partial c}{\partial x}\left\{\frac{(n-l+2)(l-1)\,l}{(2l-1)(2l-3)}b^{(0)}_{n,l-2}\right.$$

$$+\left[\frac{(n-l)(l+1)^2}{(2l+1)(2l+3)}+\frac{(n+l+1)\,l^2}{(2l-1)(2l+1)}\right]b^{(0)}_{nl}$$

$$\left.+\frac{(l+2)(l+1)(n+l+3)}{(2l+3)(2l+5)}b^{(0)}_{n,l+2}\right\}$$

$$-\frac{(n-l+2)\,l}{(n+1)(2l-1)}\frac{1}{\rho}\frac{\partial\rho}{\partial x}b^{(0)}_{n+1,l-1}$$

$$-\frac{(l+1)(n+l+3)}{(n+1)(2l+3)}\frac{1}{\rho}\frac{\partial}{\partial x}(\rho b^{(0)}_{n+1,l+1}) \tag{29}$$

If one puts $\beta^{(0,1)}_{nl}=0$, which is equivalent to Grad's expansion, the right-hand side of Eq. (28) may be considered as the inhomogeneous terms of the differential equation for $b^{(1)}_{nl}$. Considering $b^{(0)}_{nl}\sim e^{-\lambda_{nl}\rho\tau}$, we know that the right-hand side of Eq. (28) will contain a term

$$-c\frac{\partial b^{(0)}_{nl}}{\partial x}\sim\tau\lambda_{nl}c\frac{\partial\rho}{\partial x}e^{-\lambda_{nl}\rho\tau}$$

and $b^{(1)}_{nl}$ will contain a term

$$\frac{1}{2}\tau^2\left(\lambda_{nl}c\frac{\partial\rho}{\partial x}\right)e^{-\lambda_{nl}\rho\tau}$$

At the next step $b^{(2)}_{nl}$ will contain a term

$$\frac{1}{2}\frac{\tau^4}{4}\left(\lambda_{nl}c\frac{\partial\rho}{\partial x}\right)^2e^{-\lambda_{nl}\rho\tau}$$

and it can easily be inferred that $b^{(j)}_{nl}$ will contain

$$\frac{\tau^{2j}}{(2j)!!}\left(\lambda_{nl}c\frac{\partial\rho}{\partial x}\right)^j e^{-\lambda_{nl}\rho\tau}$$

which leads to a secular term. In fact, the sum of those terms is

$$\exp\left(-\lambda_{nl}\rho\tau+\frac{1}{2}\varepsilon\lambda_{nl}c\frac{\partial\rho}{\partial x}\tau^2\right)$$

Clearly, the absolute value of the second term in the exponent will be larger than that of the first when τ is large enough. In order to remove such secular terms, we have to choose $\beta_{nl}^{(0,1)}$ carefully.

In the case $l \neq 0$ and 1, we may put

$$\beta_{nl}^{(0,1)} = -c\, \partial b_{nl}^{(0)}/\partial x \tag{30}$$

$$\beta_{nl}^{(1,0)} + \lambda_{nl} \rho b_{nl}^{(1)} = R_{nl}^{(1)} \tag{31}$$

where

$$R_{nl}^{(1)} = -\frac{n(l+1)}{2l+3}\, \theta\, \frac{\partial b_{n-1,l+1}^{(0)}}{\partial x} - \frac{(n-l+2)\, l}{(n+1)(2l-1)}\, \frac{\partial b_{n+1,l-1}^{(0)}}{\partial x} + \tilde{\beta}_{nl}^{(1)} \tag{32}$$

If $l = 0$ or 1, $n \geqslant 4$, degeneracy occurs:

$$\lambda_{n,0} = \lambda_{n-1,1} \tag{33}$$

We may put

$$\beta_{n,0}^{(0,1)} = -c\, \frac{\partial b_{n,0}^{(0)}}{\partial x} - \frac{n}{3}\, \theta\, \frac{\partial b_{n-1,1}^{(0)}}{\partial x} - \frac{n}{6}\, \frac{\partial \theta}{\partial x}\, b_{n-1,1}^{(0)} \tag{34}$$

$$\beta_{n-1,1}^{(0,1)} = -c\, \frac{\partial b_{n-1,1}^{(0)}}{\partial x} - \frac{\partial b_{n,0}^{(0)}}{\partial x} - \frac{1}{2\theta} \left(\frac{\partial \theta}{\partial t} + c\, \frac{\partial \theta}{\partial x} \right) b_{n-1,1}^{(0)} \tag{35}$$

The last terms in the above equations are introduced for convenience in solving $b_{n,0}^{(0)}$ and $b_{n-1,1}^{(0)}$ (see below). From Eqs. (28), (34), and (35) we obtain

$$\beta_{n,0}^{(1,0)} + \lambda_{n,0} \rho b_{n,0}^{(1)} = R_{n,0}^{(1)} \tag{36}$$

$$\beta_{n-1,1}^{(1,0)} + \lambda_{n,0} \rho b_{n-1,1}^{(1)} = R_{n-1,1}^{(1)} \tag{37}$$

where

$$R_{n,0}^{(1)} = \frac{n}{6}\, \frac{\partial \theta}{\partial x}\, b_{n-1,1}^{(0)} + \tilde{\beta}_{n,0}^{(1)} \tag{38}$$

$$R_{n-1,1}^{(1)} = -\frac{2(n-1)}{5}\, \theta\, \frac{\partial b_{n-2,2}^{(0)}}{\partial x} + \frac{1}{2\theta} \left(\frac{\partial \theta}{\partial t} + c\, \frac{\partial \theta}{\partial x} \right) b_{n-1,1}^{(0)} + \tilde{\beta}_{n-1,1}^{(1)} \tag{39}$$

4. HIGHER ORDER APPROXIMATIONS

The higher order approximation to Eq. (23) is $(j \geqslant 2)$

$$\left(\frac{\partial \psi}{\partial t}\right)_j - \rho I_{\mathrm{M}}(\psi^{(j)})$$
$$= \rho \sum_{i=0}^{j-1} [J_{\mathrm{M}}(\psi^{(j)}, \xi^{(j-i)}) + J_{\mathrm{M}}(\xi^{(j-i)}, \psi^{(i)})]$$
$$+ \rho \sum_{i=0}^{j} J_{\mathrm{M}}(\psi^{(i)}, \psi^{(j-i)}) - [D_{01}\psi^{(j-2)} + D_{00}\psi^{(j-1)}$$
$$+ D_1(\psi^{(j-1)}) + D_2(\psi^{(j-1)})] \tag{40}$$

the e_{nl} component of which is

$$\sum_{i=0}^{j} \beta_{nl}^{(i,j-i)} + \lambda_{nl}\rho b_{nl}^{(j)}$$
$$= -\frac{n(l+1)}{2l+3}\theta \frac{\partial b_{n-1,l+1}^{(j-1)}}{\partial x} - c\frac{\partial b_{nl}^{(j-1)}}{\partial x}$$
$$-\frac{(n-l+2)\,l}{(n+1)(2l-1)} \frac{\partial b_{n+1,l-1}^{(j-1)}}{\partial x} + \tilde{\beta}_{nl}^{(j)} \tag{41}$$

where

$$\tilde{\beta}_{nl}^{(j)} = 2\rho \sum_{i=0}^{j-1} \sum_{n'l'n''l''} b_{n'l'}^{(i)} a_{n''l''}^{(j-i)} h_{\mathrm{M}nl}^{n'l'n''l''}$$
$$+ \rho \sum_{i=0}^{j} \sum_{n'l'n''l''} b_{n'l'}^{(j)} b_{n''l''}^{(j-i)} h_{\mathrm{M}nl}^{n'l'n''l''}$$
$$- nq_1\left[\frac{l}{2l-1} b_{n-1,l-1}^{(j-2)} + \frac{l+1}{2l+3} b_{n-1,l+1}^{(j-2)}\right] - \frac{1}{2}n(n-1)\,s_1 b_{n-2,l}^{(j-2)}$$
$$-\frac{1}{2}n(n-1)(n-2)\,\theta\frac{\partial\theta}{\partial x}\left[\frac{l}{2l-1} b_{n-3,l-1}^{(j-1)} + \frac{l+1}{2l+3} b_{n-3,l+1}^{(j-1)}\right]$$
$$-n(n-1)\,\theta\frac{\partial c}{\partial x}\left[\frac{l(l-1)}{(2l-1)(2l-3)} b_{n-2,l-2}^{(j-1)} + \frac{2l(l+1)}{3(2l-1)(2l+3)} b_{n-2,l}^{(j-1)}\right.$$
$$\left. + \frac{(l+1)(l+2)}{(2l+3)(2l+5)} b_{n-2,l+2}^{(j-1)}\right]$$
$$-\frac{ln}{2l-1}\theta\frac{\partial b_{n-1,l-1}^{(j-1)}}{\partial x} - \frac{n}{2}\frac{\partial\theta}{\partial x}\left[\frac{l(n-l)}{2l-1} b_{n-1,l-1}^{(j-1)}\right.$$

$$+\frac{(l+1)(n+l+1)}{2l+3}\,b^{(j-1)}_{n-1,l+1}\Bigg]$$

$$-\frac{\partial c}{\partial x}\left\{\frac{(n-l+2)(l-1)\,l}{(2l-1)(2l-3)}\,b^{(j-1)}_{n,l-2}\right.$$

$$+\left[\frac{(n-l)(l+1)^2}{(2l+1)(2l+3)}+\frac{(n+l+1)\,l^2}{(2l-1)(2l+1)}\right]b^{(j-1)}_{nl}$$

$$\left.+\frac{(l+2)(l+1)(n+l+3)}{(2l+3)(2l+5)}\,b^{(j-1)}_{n,l+2}\right\}$$

$$-\frac{(n-l+2)\,l}{(n+1)(2l-1)}\frac{1}{\rho}\frac{\partial\rho}{\partial x}\,b^{(j-1)}_{n+1,l-1}$$

$$-\frac{(l+1)(n+l+3)}{(n+1)(2l+3)}\frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho b^{(j-1)}_{n+1,l+1}\right) \tag{42}$$

In order to remove secular terms, we put

$$\beta^{(0,j)}_{nl}=\beta^{(1,j-1)}_{nl}=\cdots=\beta^{(j-2,2)}_{nl}=0 \tag{43}$$

In addition, we put

$$\beta^{(j-1,1)}_{nl}=-c\,\frac{\partial b^{(j-1)}_{nl}}{\partial x} \tag{44}$$

$$\beta^{(j,0)}_{nl}+\lambda_{nl}\rho b^{(j)}_{nl}=R^{(j)}_{nl} \tag{45}$$

$$R^{(j)}_{nl}=-\frac{n(l+1)}{2l+3}\,\theta\,\frac{\partial b^{(j-1)}_{n-1,l+1}}{\partial x}-\frac{l(n-l+2)}{(n+1)(2l-1)}\frac{\partial b^{(j-1)}_{n+1,l-1}}{\partial x}+\tilde{\beta}^{(j)}_{nl} \tag{46}$$

for $l\neq 0$ and 1; and put

$$\beta^{(j-1,1)}_{n,0}=-c\,\frac{\partial b^{(j-1)}_{n,0}}{\partial x}-\frac{n}{3}\,\theta\,\frac{\partial b^{(j-1)}_{n-1,1}}{\partial x}-\frac{n}{6}\frac{\partial\theta}{\partial x}\,b^{(j-1)}_{n-1,1} \tag{47}$$

$$\beta^{(j-1,1)}_{n-1,1}=-c\,\frac{\partial b^{(j-1)}_{n-1,1}}{\partial x}-\frac{b^{(j-1)}_{n,0}}{\partial x}-\frac{1}{2\theta}\left(\frac{\partial\theta}{\partial t}+c\,\frac{\partial\theta}{\partial x}\right)b^{(j-1)}_{n-1,1} \tag{48}$$

$$\beta^{(j,0)}_{n,0}+\lambda_{n0}\rho b^{(j)}_{n0}=R^{(j)}_{n0} \tag{49}$$

$$\beta^{(j,0)}_{n-1,1}+\lambda_{n0}\rho b^{(j)}_{n-1,1}=R^{(j)}_{n-1,1} \tag{50}$$

for $l=0$ or 1, $n \geqslant 4$, where

$$R^{(j)}_{n,0} = \frac{n}{6} \frac{\partial \theta}{\partial x} b^{(j-1)}_{n-1,1} + \tilde{\beta}^{(j)}_{n,0}$$
$$R^{(j)}_{n-1,1} = -\frac{2(n-1)}{5} \theta \frac{\partial b^{(j-1)}_{n-2,2}}{\partial x} + \frac{1}{2\theta}\left(\frac{\partial \theta}{\partial t} + c \frac{\partial \theta}{\partial x}\right) b^{(j-1)}_{n-1,1} + \tilde{\beta}^{(j)}_{n-1,1} \tag{51}$$

Thus, Eq. (21) becomes

$$\frac{\partial b^{(j)}_{n,l}}{\partial t} = \beta^{(j,0)}_{nl} + \varepsilon \beta^{(j,1)}_{nl}, \qquad j = 0, 1, \ldots \tag{52}$$

which is an exact equation for $b^{(j)}_{nl}$.

Solving Eq. (52) is not very difficult. If $l \neq 0$ and 1, Eq. (52) becomes

$$\frac{\partial b^{(j)}_{nl}(x, \tau)}{\partial \tau} + \varepsilon c(x, \tau) \frac{\partial b^{(j)}_{nl}(x, \tau)}{\partial x} + \lambda_{nl} \rho(x, \tau)\, b^{(j)}_{nl}(x, \tau) = R^{(j)}_{nl} \tag{53}$$

The characteristic curves of Eq. (53) are determined by

$$dx/d\tau = \varepsilon c(x, \tau) \tag{54}$$

Denoting $x(\tau = 0) = x_0$, we obtain from Eq. (54) that $x = x(x_0, \tau)$ for a given x_0, and the inverse function $x_0 = x_0(x, \tau)$ follows. Hence, the solution of Eq. (53) is

$$b^{(j)}_{nl}(x, \tau) = b^{(j)}_{nl}(x_0, 0)\, e^{-\gamma_{nl}} + e^{-\gamma_{nl}} \int_0^\tau e^{\gamma'_{nl}} R^{(j)}_{nl}(x(x_0, \tau'), \tau')\, d\tau' \tag{55}$$

where

$$\gamma_{nl}(x_0, \tau) = \lambda_{nl} \int_0^\tau \rho(x(x_0, \tau'), \tau')\, d\tau' \tag{56}$$

$$\gamma'_{nl} \equiv \gamma_{nl}(x_0, \tau') = \lambda_{nl} \int_0^{\tau'} \rho(x(x_0, \tau''), \tau'')\, d\tau'' \tag{57}$$

If $l=0$ or 1, $n \geqslant 4$, Eq. (52) becomes

$$\frac{\partial b^{(j)}_{n,0}}{\partial \tau} + \varepsilon c \frac{\partial b^{(j)}_{n,0}}{\partial x} + \frac{\varepsilon n}{3} \theta \frac{\partial b^{(j)}_{n-1,1}}{\partial x} + \frac{\varepsilon n}{6} \frac{\partial \theta}{\partial x} b^{(j)}_{n-1,1} + \lambda_{n0} \rho b^{(j)}_{n0} = R^{(j)}_{n,0} \tag{58a}$$

or

$$\frac{\partial b^{(j)}_{n-1,1}}{\partial \tau} + \varepsilon c \frac{\partial b^{(j)}_{n-1,1}}{\partial x} + \varepsilon \frac{\partial b^{(j)}_{n,0}}{\partial x}$$
$$+ \frac{\varepsilon}{2\theta}\left(\frac{\partial \theta}{\partial t} + c \frac{\partial \theta}{\partial x}\right) b^{(j)}_{n-1,1} + \lambda_{n0} \rho b^{(j)}_{n-1,1} = R^{(j)}_{n-1,1} \tag{58b}$$

Denoting

$$b_{n-}^{(j)} = b_{n0}^{(j)} - \left(\frac{n\theta}{3}\right)^{1/2} b_{n-1,1}^{(j)}, \qquad b_{n+}^{(j)} = b_{n0}^{(j)} + \left(\frac{n\theta}{3}\right)^{1/2} b_{n-1,1}^{(j)} \tag{59}$$

we can rewrite Eqs. (58) as two independent equations:

$$\frac{\partial b_{n-}^{(j)}}{\partial \tau} + \varepsilon \left[c - \left(\frac{n\theta}{3}\right)^{1/2} \right] \frac{\partial b_{n-}^{(j)}}{\partial x} + \lambda_{n0} \rho b_{n-}^{(j)} = R_{n-}^{(j)} \tag{60}$$

$$\frac{\partial b_{n+}^{(j)}}{\partial \tau} + \varepsilon \left[c + \left(\frac{n\theta}{3}\right)^{1/2} \right] \frac{\partial b_{n+}^{(j)}}{\partial x} + \lambda_{n0} \rho b_{n+}^{(j)} = R_{n+}^{(j)} \tag{61}$$

where

$$R_{n-}^{(j)} = R_{n0}^{(j)} - \left(\frac{n\theta}{3}\right)^{1/2} R_{n-1,1}^{(j)}, \qquad R_{n-}^{(j)} = R_{n0}^{(j)} + \left(\frac{n\theta}{3}\right)^{1/2} R_{n-1,1}^{(j)} \tag{62}$$

Equations (60) and (61) can be solved exactly in a way similar to solving Eq. (53). The characteristic curves of Eq. (60) are given by

$$\frac{dx(x_0^-, \tau)}{d\tau} = \varepsilon \left[c - \left(\frac{n\theta}{3}\right)^{1/2} \right], \qquad x(x_0^-, 0) = x_0^- \tag{63}$$

If $x_0^- = x_0^-(x, \tau)$ is determined from the solution of Eq. (63), the solution of Eq. (60) will be

$$b_{n-}^{(j)} = b_{n-}^{(j)}(x_0^-, 0)\, e^{-\gamma_{n-}} + e^{-\gamma_{n-}} \int_0^\tau e^{\gamma'_{n-}} R_{n-}^{(j)}(x(x_0^-, \tau'), \tau')\, d\tau' \tag{64}$$

where

$$\gamma_{n-} = \lambda_{n0} \int_0^\tau \rho(x(x_0^-, \tau'), \tau')\, d\tau' \tag{65}$$

$$\gamma'_{n-} \equiv \gamma_{n-}(x_0^-, \tau') = \lambda_{n0} \int_0^{\tau'} \rho(x(x_0^-, \tau''), \tau'')\, d\tau'' \tag{66}$$

The characteristic curves of Eq. (62) are given by

$$\frac{dx(x_0^+, \tau)}{d\tau} = \varepsilon \left[c + \left(\frac{n\theta}{3}\right)^{1/2} \right], \qquad x(x_0^+, 0) = x_0^+ \tag{67}$$

If $x_0^+ = x_0^+(x_0, \tau)$ is determined from the solution of Eq. (67), the solution of Eq. (61) will be

$$b_{n+}^{(j)} = b_{n+}^{(j)}(x_0^+, 0)\, e^{-\gamma_{n+}} + e^{-\gamma_{n+}} \int_0^\tau e^{\gamma'_{n+}} R_{n+}^{(j)}(x(x_0^+, \tau'), \tau')\, d\tau' \tag{68}$$

where

$$\gamma_{n+} = \lambda_{n0} \int_0^\tau \rho(x(x_0^+, \tau'), \tau')\, d\tau' \tag{69}$$

$$\gamma'_{n+} \equiv \gamma_{n+}(x_0^+, \tau') = \lambda_{n0} \int_0^{\tau'} \rho(x(x_0^+, \tau''), \tau'')\, d\tau'' \tag{70}$$

Finally, we obtain from Eqs. (59) that

$$b_{n,0}^{(j)} = \frac{1}{2}(b_{n-}^{(j)} + b_{n+}^{(j)}), \qquad b_{n-1,1}^{(j)} = \frac{1}{2}\left(\frac{3}{n\theta}\right)^{1/2} (b_{n+}^{(j)} - b_{n-}^{(j)}) \tag{71}$$

It is very easy to see from Eqs. (63) and (67) that the high-moment sound waves do appear in a Maxwell gas and propagate with speed $(n\theta/3)^{1/2}$. But they decay very rapidly because of

$$\lambda_{n0} = \lambda_{n-1,1} > 0$$

This fact was noticed by Wang Chang and Uhlenbeck.[4] As mentioned in the introduction, in Grad's solution the high-moment sound waves were missed, because the secular terms were not removed.

5. THE CONNECTION WITH THE NORMAL SOLUTION

According to Eq. (12), we know that $b_{nl}^{(j)}$ should satisfy

$$\lim_{\tau \to \infty} b_{nl}^{(j)}(x, \tau) = 0 \tag{72}$$

It is evident that Eq. (72) is satisfied for $(n, l) = (2, 2), (3, 1), \ldots$, because of the factor $\exp(-\lambda \int_0^\tau \rho\, d\tau')$. It is seen from Eqs. (16) and (19) that

$$b_{nl}^{(j)}(x, 0) = -a_{nl}^{(j)}(x, 0), \qquad (n, l) = (0, 0), (1, 1), \ldots; \quad j = 1, 2, \ldots \tag{73}$$

The initial values of ρ, c, and θ are given in the normal solution. As soon as values of $a_{nl}^{(j)}(x, 0)$ for $(n, l) = (0, 0), (1, 1), (2, 0)$ are given, that of all other $a_{nl}^{(j)}(x, 0)$ will be defined as explained in Ref. 1, and $b_{nl}^{(j)}(x, 0)$ for $(n, l) = (2, 2), (3, 1), \ldots$, will be determined by Eq. (73).

The situation is different for the first three moments $(n, l) = (0, 0)$, $(1, 1)$, $(2, 0)$. Let us consider Eq. (28) as an example. In view of

$$\lambda_{00} = \lambda_{11} = \lambda_{20} = 0$$

the secular term would appear if $\beta_{nl}^{(1,0)}$ contained a term without any factor like $\exp(-\lambda \int_0^\tau \rho\, d\tau')$. (For instance, if $\beta_{nl}^{(1,0)}$ contained a term that remained constant, $b_{nl}^{(1)}$ would contain a term proportional to τ.) But the right-hand side of Eq. (28) has no such term, so we may put

$$\beta_{00}^{(0,1)} = \beta_{11}^{(0,1)} = \beta_{20}^{(0,1)} = 0 \tag{74}$$

Similarly, we may retain Eq. (43) and put

$$\beta_{00}^{(j-1,1)} = \beta_{11}^{(j-1,1)} = \beta_{20}^{(j-1,1)} = 0, \qquad j = 1, 2,... \tag{75}$$

Then Eq. (21) becomes

$$\frac{\partial b_{nl}^{(j)}}{\partial \tau} = \beta_{nl}^{(j,0)}, \qquad (n, l) = (0, 0), (1, 1), (2, 0); \quad j = 0, 1,... \tag{76}$$

In the case $j = 0$, we know from Eq. (5) that $b_{00}^{(0)}(x, 0) = b_{11}^{(0)}(x, 0) = b_{20}^{(0)}(x, 0) = 0$. We obtain from Eqs. (25) and (76) that

$$b_{nl}^{(0)}(x, \tau) = 0, \qquad (n, l) = (0, 0), (1, 1), (2, 0) \tag{77}$$

In the case $j = 1$, in virtue of Eqs. (28) and (75), we may write Eq. (76) as

$$\begin{gathered} \frac{\partial b_{00}^{(1)}}{\partial \tau} = 0, \qquad \frac{\partial b_{11}^{(1)}}{\partial \tau} = -\frac{1}{\rho} \frac{\partial}{\partial x} (\rho b_{22}^{(0)}), \\ \frac{\partial b_{20}^{(1)}}{\partial \tau} = \frac{-5}{9\rho} \frac{\partial}{\partial x} (\rho b_{31}^{(0)}) - \frac{2}{3} \frac{\partial c}{\partial x} b_{22}^{(0)} \end{gathered} \tag{78}$$

Hence, using Eqs. (78) and (73), we get

$$\begin{aligned} b_{00}^{(1)}(x, \tau) &= -a_{00}^{(1)}(x, 0) \\ b_{11}^{(1)}(x, \tau) &= -a_{11}^{(1)}(x, 0) - \int_0^\tau \frac{1}{\rho} \frac{\partial}{\partial x} (\rho b_{22}^{(0)})\, d\tau' \\ b_{20}^{(1)}(x, \tau) &= -a_{20}^{(1)}(x, 0) - \int_0^\tau \frac{5}{9\rho} \frac{\partial}{\partial x} (\rho b_{31}^{(0)})\, d\tau' - \frac{2}{3} \int_0^\tau \frac{\partial c}{\partial x} b_{22}^{(0)}\, d\tau' \end{aligned} \tag{79}$$

According to Eq. (72), we have to put

$$\begin{aligned} a_{00}^{(1)}(x, 0) &= 0 \\ a_{11}^{(1)}(x, 0) &= -\int_0^\infty \frac{1}{\rho} \frac{\partial}{\partial x} (\rho b_{22}^{(0)})\, d\tau' \\ a_{20}^{(1)}(x, 0) &= -\frac{5}{9} \int_0^\infty \frac{1}{\rho} \frac{\partial}{\partial x} (\rho b_{31}^{(0)})\, d\tau' - \frac{2}{3} \int_0^\infty \frac{\partial c}{\partial x} b_{22}^{(0)}\, d\tau' \end{aligned} \tag{80}$$

In the case $j \geqslant 2$, the procedure is similar. It is seen from Eqs. (41), (43), and (75) that

$$\beta_{00}^{(j,0)} = -c\frac{\partial b_{00}^{(j-1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho b_{11}^{(j-1)})$$

$$\beta_{11}^{(j,0)} = -c\frac{\partial b_{11}^{(j-1)}}{\partial x} - \frac{\partial b_{20}^{(j-1)}}{\partial x} - q_1 b_{00}^{(j-2)} - \theta\frac{\partial b_{00}^{(j-1)}}{\partial x} - \frac{\partial c}{\partial x} b_{11}^{(j-1)}$$

$$-\frac{1}{\rho}\frac{\partial \rho}{\partial x} b_{20}^{(j-1)} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho b_{22}^{(j-1)}) \tag{81}$$

$$\beta_{20}^{(j,0)} = -\frac{2}{3}\theta\frac{\partial b_{11}^{(j-1)}}{\partial x} - c\frac{\partial b_{20}^{(j-1)}}{\partial x} - \frac{2}{3} q_1 b_{11}^{(j-2)} - s_1 b_{00}^{(j-2)} - \frac{\partial \theta}{\partial x} b_{11}^{(j-1)}$$

$$-\frac{2}{3}\frac{\partial c}{\partial x}(b_{20}^{(j-1)} + b_{22}^{(j-1)}) - \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho b_{31}^{(j-1)})$$

Hence we obtain that the solution of Eq. (76) under the condition (72) is

$$b_{nl}^{(j)}(x, \tau) = -\int_{\tau}^{\infty} \beta_{nl}^{(j,0)}\, d\tau' \tag{82}$$

$$(n, l) = (0, 0), (1, 1), (2, 0); \quad j = 2, 3, \ldots,$$

and the initial value of $a_{nl}^{(j)}(x, t)$ for $(n, l) = (0, 0), (1, 1), (2, 0)$ is

$$a_{nl}^{(j)}(x, 0) = \int_{0}^{\infty} \beta_{nl}^{(j,0)}\, d\tau' \tag{83}$$

$$(n, l) = (0, 0), (1, 1), (2, 0); \quad j = 2, 3, \ldots$$

We mentioned in Ref. 1 that several coefficients in the normal solution, such as $a_{nl}^{(j)}(x, 0)$ for $(n, l) = (0, 0), (1, 1), (2, 0)$, $j = 1, 2, 3, \ldots$, remained undetermined. Now we find that these coefficients are determined by Eqs. (80) and (83) in order to satisfy the condition (72). In other words, if we take initial values of $a_{nl}^{(j)}(x, t)$ that do not agree with Eqs. (80) and (83), the solution φ will not tend to φ_n outside the initial layer. This is another kind of secular term, which can cause the initial layer solution to be invalid. In order to avoid this type of secular term, we are forced to expand the hydrodynamic variables in a power series in ε. In other words, we need the counterparts of $b_{00}^{(j)}$, $b_{11}^{(j)}$, and $b_{20}^{(j)}$ $(j = 1, 2, \ldots)$ in the normal solution. However, we cannot find the counterparts for $j = 1, 3, 5, \ldots$, in Cercignani's expansion, since the hydrodynamic variables there are expanded in powers of ε^2 even though each term is dependent on ε. As mentioned in the introduction, this is the second difference between Cercignani's normal solution and that presented in Ref. 1.

Finally, we summarize our procedure as follows:

1. Find $b_{00}^{(0)}(x, \tau)$, $b_{11}^{(0)}(x, \tau)$, and $b_{20}^{(0)}(x, \tau)$ from Eqs. (77), and evaluate other $b_{nl}^{(0)}(x, \tau)$ from Eqs. (55), (64), (68), and (71) for $j=0$.

2. Evaluate $b_{00}^{(1)}(x, \tau)$, $b_{11}^{(1)}(x, \tau)$, and $b_{10}^{(1)}(x, \tau)$ from Eqs. (79), and other $b_{nl}^{(1)}(x, \tau)$ from Eqs. (55), (64), (68), and (71), and the initial values of $a_{00}^{(1)}(x, t)$, $a_{11}^{(1)}(x, t)$, and $a_{20}^{(1)}(x, t)$ from Eqs. (80) for $j=1$.

3. Evaluate $b_{00}^{(j)}(x, \tau)$, $b_{11}^{(j)}(x, \tau)$, and $b_{20}^{(j)}(x, \tau)$ from Eqs. (82) and (81), other $b_{nl}^{(j)}(x, \tau)$ from Eqs. (55), (64), (68), and (71), and the initial values $a_{00}^{(j)}(x, t)$, $a_{11}^{(j)}(x, t)$, and $a_{20}^{(j)}(x, t)$ from Eqs. (83) and (81) for $j \geqslant 2$.

4. Evaluate

$$a_{nl}(x, t) = \sum_{j=0}^{\infty} \left[a_{nl}^{(j)}(x, t) + b_{nl}^{(j)}(x, t)\right] \varepsilon^j$$

which determines the hydrodynamic variables, by the formulas given in Ref. 1.

6. EXTENSION TO NON-MAXWELL MOLECULES

The situation for non-Maxwell molecules is more complicated. The fact that in the preceding paragraphs we could work our way recursively through the initial layer to an *arbitrary* initial condition was due to the special simplifying features of the Maxwell model. In order to get explicit results for non-Maxwell molecules, we must restrict ourselves to initial conditions sufficiently close to a local Maxwell distribution that linearization makes sense. In that case, the discussion can be made quite similar to that above.

The real, symmetric operator I in Eq. (18) can be diagonalized

$$I(d_{nl}) = -\Lambda_{nl}\, d_{nl} \tag{84}$$

where

$$d_{nl} = \sum_{n'} D_{nl}^{n'l} e_{n'l} = \sum_{n'} D_{nl}^{n'l} \frac{(-ik)^{n'}}{n'!} P_l(\mu) \tag{85}$$

In general, the eigenvalues Λ_{nl} and the eigenvectors d_{nl} are dependent on θ. We know from Eqs. (111) in Ref. 1 that

$$d_{00} = e_{00}, \qquad d_{11} = e_{11}, \qquad d_{20} = e_{20}, \qquad \Lambda_{00} = \Lambda_{11} = \Lambda_{20} = 0 \tag{86}$$

All other eigenvalues $\Lambda_{nl} > 0$. We assume that there is no more degeneracy in the spectrum of the operator I.

Letting

$$\psi = \sum_{j=0}^{\infty} \varepsilon^j \psi^{(j)} = \sum_{j=0}^{\infty} \varepsilon^j \sum_{nl} B_{nl}^{(j)}(x, \tau)\, d_{nl} \tag{87}$$

$$\frac{\partial B_{nl}^{(j)}}{\partial \tau} = \beta_{nl}^{(j,0)} + \varepsilon \beta_{nl}^{(j,1)} \tag{88}$$

we obtain from Eq. (18) that the d_{nl} component of the ε^0-order approximation is

$$\beta_{nl}^{(0,0)} + \Lambda_{nl} \rho B_{nl}^{(0)} = \rho \sum_{n'l'n''l''} B_{n'l'}^{(0)} B_{n''l''}^{(0)} H_{nl}^{n'l'n''l''} \tag{89}$$

where $H_{nl}^{n'l'n''l''}$ are defined by

$$\frac{1}{2}\,[J'(d_{n'l'}, d_{n''l''}) + J'(d_{n''l''}, d_{n'l'})] = \sum_{nl} H_{nl}^{n'l'n''l''}\, d_{nl} \tag{90}$$

It is easy to get

$$\begin{aligned} &H_{00}^{n'l'n''l''} = H_{11}^{n'l'n''l''} = H_{20}^{n'l'n''l''} = 0, \qquad H_{nl}^{n'l'n''l''} = H_{nl}^{n''l''n'l'} \\ &H_{nl}^{n'l'n''l''} = 0 \qquad \text{if} \quad |l' - l''| > l \quad \text{or} \quad l > l' + l'' \end{aligned} \tag{91}$$

The d_{nl} component of the ε-order approximation to Eq. (18) is

$$\begin{aligned} &\beta_{nl}^{(1,0)} + \beta_{nl}^{(0,1)} + \Lambda_{nl} \rho B_{nl}^{(1)} \\ &\quad = 2\rho \sum_{n'l'n''l''} B_{n'l'}^{(0)} (A_{n''l''}^{(1)} + B_{n''l''}^{(1)})\, H_{nl}^{n'l'n''l''} \\ &\qquad - [d_{nl}, D_m \psi^{(0)}] - C \frac{\partial B_{nl}^{(0)}}{\partial x} \end{aligned} \tag{92}$$

where $[d_{nl}, \psi]$ denotes the coefficient of d_{nl} in ψ, $A_{nl}^{(1)}$ denotes the coefficient of d_{nl} in $\xi^{(j)}$, and

$$D_m \equiv D_{00} + D_2 + \rho e_{11} \frac{\partial}{\partial x} + i\mu \frac{\partial^2}{\partial k\, \partial x} + \frac{i(1-\mu^2)}{k} \frac{\partial^2}{\partial \mu\, \partial x} \tag{93}$$

The difference between the non-Maxwell gas and the Maxwell gas is that closed equations for $B_{nl}^{(0)}$ cannot be obtained even though the expression of $\beta_{nl}^{(0,1)}$ in Eqs. (88) and (89) is determined, because the $H_{nl}^{n'l'n''l''}$

do not necessarily vanish when $n < n' + n''$. But if the deviation of the initial distribution from a local Maxwellian distribution is small, for example, if ψ_0 is $O(\varepsilon)$,

$$\psi_0 = \varepsilon \psi_{01}$$

we can obtain from Eqs. (4) and (5) that

$$\begin{aligned} &\psi^{(0)}(\tau = 0) = 0 \\ &\psi^{(1)}(\tau = 0) = \psi_{01} - \xi^{(1)}(t = 0) \\ &\psi^{(j)}(\tau = 0) = -\xi^{(j)}(t = 0), \qquad j \geqslant 2 \end{aligned}$$

Taking $\beta_{nl}^{(0,1)} = 0$, we get

$$\psi^{(0)} \equiv 0 \tag{94}$$

and we get from Eq. (92) that

$$\beta_{nl}^{(1,0)} + \Lambda_{nl} \rho B_{nl}^{(1)} = 0 \tag{95}$$

The d_{nl} components of the ε^j-order approximation to Eq. (18) for $j \geqslant 2$ are

$$\beta_{nl}^{(j,0)} + \beta_{nl}^{(j-1,1)} + \Lambda_{nl} \rho B_{nl}^{(j)} = -c \frac{\partial B_{nl}^{(j-1)}}{\partial x} + R_{nl}^{(j)} \tag{96}$$

where

$$\begin{aligned} R_{nl}^{(j)} = 2\rho \sum_{i=1}^{j-1} \sum_{n'l'n''l''} B_{n'l'}^{(i)} A_{n''l''}^{(j-i)} H_{nl}^{n'l'n''l''} \\ + \rho \sum_{i=1}^{j-1} \sum_{n'l'n''l''} B_{n'l'}^{(i)} B_{n''l''}^{(j-i)} H_{nl}^{n'l'n''l''} \\ - [d_{nl}, D_m \psi^{(j-1)}] - [d_{nl}, D_{01} \psi^{(j-2)}] \end{aligned} \tag{97}$$

In the case $(n, l) \neq (0, 0)$, $(1, 1)$, $(2, 0)$, letting

$$\beta_{nl}^{(j-1,1)} = -c \frac{\partial B_{nl}^{(j-1)}}{\partial x} \tag{98}$$

$$\beta_{nl}^{(j,0)} + \Lambda_{nl} \rho B_{nl}^{(j)} = R_{nl}^{(j)} \tag{99}$$

we obtain from Eq. (88) that

$$\frac{\partial B_{nl}^{(j)}}{\partial \tau} + \varepsilon c \frac{\partial B_{nl}^{(j)}}{\partial x} + \Lambda_{nl} \rho B_{nl}^{(j)} = R_{nl}^{(j)} \tag{100}$$

The solution of Eq. (100) is

$$B_{nl}^{(j)} = B_{nl}^{(j)}(x_0, 0)\, e^{-\gamma_{nl}} + e^{-\gamma_{nl}} \int_0^{\tau} e^{\gamma'_{nl}} R_{nl}^{(j)}(x(x_0, \tau'), \tau')\, dt' \tag{101}$$

where

$$\gamma_{nl} = \Lambda_{nl} \int_0^{\tau} \rho(x(x_0, \tau'), \tau')\, dt'$$

$$\gamma'_{nl} \equiv \gamma_{nl}(x_0, \tau') = \Lambda_{nl} \int_0^{\tau'} \rho(x(x_0, \tau''), \tau'')\, d\tau''$$

Here $x_0 = x_0(x, \tau)$ is determined by the following equation:

$$\frac{dx(x_0, \tau)}{d\tau} = \varepsilon c, \qquad x(x_0, 0) = x_0$$

In the case $(n, l) = (0, 0)$, $(1, 1)$, $(2, 0)$, we have $A_{nl}^{(j)} = a_{nl}^{(j)}$. Taking $\beta_{nl}^{(j-1,1)} = 0$, we get from Eq. (88) that

$$\frac{\partial B_{nl}^{(j)}}{\partial \tau} = \beta_{nl}^{(j,0)}, \qquad (n, l) = (0, 0), (1, 1), (2, 0)$$

where

$$\beta_{nl}^{(j,0)} = R_{nl}^{(j)} - c\, \frac{\partial B_{nl}^{(j-1)}}{\partial x}, \qquad (n, l) = (0, 0), (1, 1), (2, 0)$$

Hence, we have

$$B_{nl}^{(j)}(x, \tau) = B_{nl}^{(j)}(x, 0) + \int_0^{\tau} \beta_{nl}^{(j,0)}\, d\tau' \qquad (n, l) = (0, 0), (1, 1), (2, 0) \tag{102}$$

Considering the condition (12), we know that

$$B_{nl}^{(j)}(x, \tau) = -\int_{\tau}^{\infty} \beta_{nl}^{(j,0)}\, d\tau', \qquad (n, l) = (0, 0), (1, 1), (2, 0) \tag{103}$$

and get the initial values $A_{nl}^{(j)}(x, 0)$ for $(n, l) = (0, 0)$, $(1, 1)$, and $(2, 0)$.

7. CONCLUDING REMARKS

We have shown that the improved initial layer solution is different from Grad's solution. We stress two main points here for distinctness.

The first is that the secular terms have been removed in the improved expansion. The higher moments of the distribution function in the dominant terms of Grad's expansion decay as $e^{-\lambda_{nl}\rho\tau}$. If we calculate all terms in higher order approximations, the series will be divergent as $t \to \infty$. In fact, we have seen such a divergent series in Section 3, namely

$$\sum_{j=0}^{\infty} \varepsilon^{j} \frac{\tau^{2j}}{(2j)!!}\left(\lambda_{nl} c \frac{\partial \rho}{\partial x}\right)^{j} \exp(-\lambda_{nl}\rho\tau) = \exp\left[-\lambda_{nl}\rho\tau + \frac{1}{2}\varepsilon\lambda_{nl} c \frac{\partial \rho}{\partial x}\tau^{2}\right]$$

But in the improved solution, the higher moments decay as $e^{-\gamma_{nl}}$. If we neglect the difference between x and x_0 in Eq. (56), we find that

$$\gamma_{nl} \approx \lambda_{nl} \int_0^{\tau} \left[\rho(x, 0) + \frac{\partial \rho}{\partial t} \varepsilon\tau' + \cdots\right] d\tau'$$

In virtue of

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho}{\partial x} = 0$$

we get

$$\gamma_{nl} \approx \lambda_{nl}\rho(x, 0)\,\tau - \frac{\varepsilon}{2}\lambda_{nl} c \frac{\partial \rho}{\partial x}\tau^{2}$$

This shows that, in our solution, the divergent series in Grad's expansion has been summed and become bounded. In his paper, Grad[(2)] noted that his expansion is asymptotic and can be used only within a finite time. Our improved solution is valid for any time, including $t \to \infty$.

The second point is that the higher moment sound waves reappear in the improved solution. Actually, this conclusion follows directly from the requirement that the secular terms in the expansion should be removed. The reason the higher moment sound waves were missed in Grad's expansion is nothing but the existence of the secular terms.

The initial layer solution can describe the relaxation of a system far from the state of equilibrium. We illustrate this by a simple example.

Suppose that an explosion in the Maxwell gas produces an initial distribution function

$$f(t=0)=(2\pi)^{3/2}\,(1-z^2)^{-1/2}\exp\left[-\frac{(v-v_0 z)^2}{2(1-z^2)}\right]$$

where

$$z=[1+\Delta+4e^{-r_c/\delta}\,\mathrm{sh}^2(r/2\delta)]^{-1}$$

$$r_c \gg \delta \gg \varepsilon, \qquad 1 \gg \Delta > 0$$

and r is the distance from the center of the explosion. Clearly, the system has spherical symmetry. The initial distribution function $f(t=0)$ tends to $(1/2\pi)\,\delta(v-v_0)$ for $r \ll r_c$ and tends to f_e for $r \gg r_c$. The condition $\Delta > 0$ ensures that the distribution function belongs to $\mathscr{H}$. Using the method of the present paper with some modifications due to the different geometry, we can get formulas quite similar to what we obtained in Sections 1–5. By means of these formulas we can calculate the dependence of the heat flux q on time τ according to the scheme indicated in the last paragraph of Section 5. For $v_0=3$, $\Delta=0.1$, $r_c=1$, and $\delta=0.1$, the results at $r=0.9926$ are as shown in Table I. Here, positive q means flux going outward from the center. Note that for very small τ, the heat flows toward the central part, where the temperature is higher. This does not contradict the second law of thermodynamics, because the increase of the entropy in the central part due to relaxation cancels out the decrease of the entropy caused by the negative heat flux and the total entropy of the system still increases. Another interesting point in this example is that a small vibration appears in the relaxation of the heat flux to the normal state. This vibration reflects the higher moment sound waves.

The results given above show the first vibration only. In fact, the second vibration is very weak because it decays rapidly. It is seen from this simple example that the relaxation process from a space-dependent state is much more complicated than that from a homogeneous state.

Table I. Dependence of Heat Flux q on Time τ

$\tau=t/\varepsilon$	0.000	0.016	0.05	0.20	0.40	0.60	1.20	2.00
q/ε	0.000	−0.13	0.36	7.48	15.47	18.87	20.29	20.25

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